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Nonlinear canonical transformations: I

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Abstract. The linear canonical transformation $Q = aq + bp$, $P = cq + dp$, ($ad - bc = 1$, $[q, p] =$ constant) is extended to include terms of the second, third and fourth degree. Products of second-degree transformations are used to construct third- and fourth-degree transformations. The question of factorisability of fourth-degree transformations is discussed.

1. Introduction

As is well known, the necessary and sufficient condition that the linear transformation

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \tag{1.1a}$$

be canonical in *either* classical *or* quantum mechanics is that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1. \tag{1.1b}$$

The transformations then form the symplectic group Sp(2). In the case of N degrees of freedom, transformations of type (1.1) form the dynamical group of the N-dimensional harmonic oscillator (Moshinsky and Quesne 1971). From the numerous applications of such transformations that have been made, we single out that of Elliott (1958) on the SU(3) symmetry of the three-dimensional oscillator in problems of nuclear structure, and also the work of Colegrave and Abdalla (1981, 1982) on time-dependent harmonic oscillators.

Turning our attention to the case of nonlinear canonical transformations, we might expect to be able to solve some new problems in quantum (or classical) mechanics provided we have suitable canonical transformations at hand. Some progress has already been made in this direction. Moshinsky *et al* (1972) have derived the dynamical group of canonical transformations for the radial oscillator problems and for the Coulomb problem with a centrifugal force of arbitrary strength. Mello and Moshinsky (1975) give further specific examples of useful transformations.

Certain anharmonic oscillator problems are obvious candidates for canonical transformation to solvable form. In this case the useful transformations would almost certainly be extensions of (1.1) to include second- and possibly higher-degree terms in q, p. This line of thought is mainly responsible for the present work. Carhart (1971) has discussed an interesting method of successive canonical transformation to treat the anharmonic oscillator. Some solvable anharmonic oscillators have been discussed

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by Flessas and Das (1980), Flessas (1981*a*, *b*) and Magyari (1981) (see also Flessas *et al* 1983).

Looking at more general transformations, Deenen (1981) has discussed non-bijective canonical transformations where the spectra of the original and transformed Hamiltonians are different. In the nonlinear case we cannot always expect a one-one correspondence between classical and quantal transformations. Halpern (1973) and Fanelli (1976) have shown that many transformations that are canonical in classical mechanics are not canonical in quantum mechanics and *vice versa*. However, this will not concern us in the work presented here, since all the transformations we shall consider may be regarded as either quantum mechanical or classical. When the occasion demands we shall indicate the quantum mechanical forms.

We shall confine our attention to transformations of the form

$$Q = f(q, p) + aq + bp, \qquad P = g(q, p) + cq + dp, \tag{1.2a, b}$$

where f, g are second- third- or fourth-degree expressions in the non-commuting variables q, p. We shall find necessary and sufficient conditions for canonicity by requiring that the following classical *or* quantum Poisson bracket relation shall hold,

$$[Q, P] = [q, p] \tag{1.3}$$

(Goldstein 1980, Wollenberg 1980).

We shall begin by considering an important class of transformations for which the degree of the functions f and g in equations (1.2) is unrestricted. We shall then particularise to the cases mentioned above. In principle it would be possible to extend our analysis to functions f and g of higher degree. Generating functions will be considered in § 6 and some applications will be given in § 7.

2. A special class of nonlinear canonical transformations

We have discovered a special class of transformations which plays a leading role in our investigation. We shall state our result in classical terms, but it is easy to see that it is equally valid in quantum mechanics, where we need to symmetrise all products with respect to q and p.

The transformation

$$Q = F(q + yp) + aq + bp, \qquad P = xF(q + yp) + cq + dp, \tag{2.1a, b}$$

$$ad - bc = 1, \tag{2.1c}$$

where F is any differentiable function of $q + yp$ and x, y are real numbers, is canonical if and only if x, y are connected by the bilinear relation

$$y = (bx - d)/(ax - c). \tag{2.1d}$$

The proof follows directly from (2.1*a*, *b*, *c*):

$$\begin{aligned}[Q, P] &= (c - ax)[F, q] + (d - bx)[F, p] + (ad - bc)[q, p] \\ &= [(ax - c)\, \partial F/\partial p - (bx - d)\, \partial F/\partial q + 1][q, p] \\ &= [q, p] \qquad \text{iff } y(ax - c) = bx - d.\end{aligned}$$

We note the following relation from (2.1*a*, *b*),

$$xQ-P=(ax-c)q+(bx-d)p, \tag{2.2a}$$

which in view of (2.1*d*) may be written

$$q+yp=(b-ay)(xQ-P). \tag{2.2b}$$

Provided $y \neq b/a$ the transformation (2.1) has the inverse

$$q=yG(xQ-P)+dQ-bP, \qquad p=-G(xQ-P)-cQ+aP, \tag{2.3a, b}$$

$$G(s)=F[(b-ay)s]/(b-ay). \tag{2.3c}$$

As we shall see very clearly later, the transformations (2.1) do *not* form a group. Thus some transformations (1.2) are 'missing' in (2.1).

2.1. Second-degree canonical transformations

We take the general second-degree transformation $T^{(2)}(a_i, b_i)$:

$$Q=a_1q^2+a_2(qp+pq)+a_3p^2+aq+bp+a_4, \tag{2.4a}$$

$$P=b_1q^2+b_2(qp+pq)+b_3p^2+cq+dp+b_4. \tag{2.4b}$$

We may without loss of generality set $a_4=b_4=0$ in the same way as we ignored translational terms in the transformation (1.1), but we note that such terms could be time dependent and so influence the generating function. It would be a trivial task to include the extra terms in such a case.

Equation (1.3) gives the following necessary and sufficient conditions for the transformation (2.4) to be canonical:

$$a_1/b_1=a_2/b_2=a_3/b_3, \qquad a_1d-b_1b=a_2c-b_2a, \tag{2.5a, b}$$

$$a_2d-b_2b=a_3c-b_3a, \qquad ad-bc=1. \tag{2.5c, d}$$

We introduce the parameters $x=b_1/a_1$ and y given by (2.1*d*); then equations (2.5*a*, *b*, *c*) reduce to

$$b_i=a_ix \qquad (i=1,2,3), \qquad a_2=a_1y, \qquad a_3=a_1y^2. \tag{2.6a,b}$$

Taking a_1 arbitrary, equations (2.4), (2.6) give

$$Q=a_1(q+yp)^2+aq+bp, \tag{2.7a}$$

$$P=a_1x(q+yp)^2+cq+dp, \tag{2.7b}$$

where a, b, c, d must satisfy (2.1*c*) and x, y must satisfy (2.1*d*). Thus all the second-degree canonical transformations (2.4) are of the form (2.1) with $F=a_1(q+yp)^2$.

3. Third-degree canonical transformations

Transformations of the type (1.2) with f and g expandable in Taylor series in the (non-commuting) variables q and p obviously form a subgroup of the complete group of all possible canonical transformations. On the other hand, transformations of the type (2.1) clearly do not form a group, so that as we increase the degree of f and g

we must eventually encounter canonical transformations that are more general than those described by equations (2.1). The first time this happens is when f and g are of the third degree.

The general third-degree transformation $T^{(3)}(a_i, b_i)$ of the form (1.2) with $[q, p] =$ constant will be written

$$Q = a_1 q^3 + 3a_2 qpq + 3a_3 pqp + a_4 p^3 + a_5 q^2 + a_6(qp + pq) + a_7 p^2 + aq + bp, \tag{3.1a}$$

$$P = b_1 q^3 + 3b_2 qpq + 3b_3 pqp + b_4 p^3 + b_5 q^2 + b_6(qp + pq) + b_7 p^2 + cq + dp, \tag{3.1b}$$

where we have used the fact that

$$q^2 p + pq^2 = 2qpq, \qquad p^2 q + qp^2 = 2pqp. \tag{3.2}$$

We write

$$\lambda = b_5/a_5, \qquad \mu = b_6/a_6, \qquad \nu = b_7/a_7; \tag{3.3}$$

then equation (1.3) leads to

$$b_i = a_i x \qquad (i = 1, 2, 3, 4), \tag{3.4a}$$

$$a_2 = a_1 y, \qquad a_3 = a_1 y^2, \qquad a_4 = a_1 y^3, \tag{3.4b}$$

$$a_6 = a_5 y(x - \lambda)/(x - \mu), \qquad a_7 = a_5 y^2(x - \lambda)/(x - \nu), \tag{3.4c}$$

$$y = \frac{b\lambda - d}{a\mu - c}\,\frac{x - \mu}{x - \lambda} = \frac{b\mu - d}{a\nu - c}\,\frac{x - \nu}{x - \mu}, \tag{3.4d}$$

$$(\lambda + \nu)(x + \mu) = 2(x\mu + \lambda\nu), \tag{3.4e}$$

$$3a_1[y(ax - c) - (bx - d)] = 2a_5^2 y(\lambda - \nu)(x - \lambda)/(x - \nu), \tag{3.4f}$$

$$ad - bc = 1. \tag{3.4g}$$

Thus the general third-degree canonical transformation is necessarily of the form

$$Q = a_1(q + yp)^3 + a_5\{q^2 + y[(x - \lambda)/(x - \mu)](qp + pq) + y^2[(x - \lambda)/(x - \nu)]p^2\} + aq + bp, \tag{3.5a}$$

$$P = a_1 x(q + yp)^3 + a_5\{\lambda q^2 + \mu y[(x - \lambda)/(x - \mu)](qp + pq) + \nu y^2[(x - \lambda)/(x - \nu)]p^2\} + cq + dp, \tag{3.5b}$$

where a, b, c, d satisfy (3.4g), a_1, x, λ are arbitrary and a_5, y, μ, ν are determined by the four equations contained in (3.4d, e, f). We note that the choice $x = \lambda$ makes $\mu = \nu = x$. (In fact the equality of any pair from x, λ, μ, ν makes them all equal.) In this case equations (3.5) take the special form (2.1) with $F = a_1(q + yp)^3 + a_5(q + yp)^2$ (note that a_5 depends on a_1 according to equation (3.4f)).

On fixing a_1, x, $\lambda\,(\neq x)$ it is surprisingly difficult at this stage to find specific solutions of equations (3.4d, e, f) for a_5, y, μ, ν. However, when we have investigated fourth-degree transformations in §4 we shall be able to construct systematically $T^{(3)}$ transformations that are *not* of the special type (2.1).

Using equation (3.4d), a useful alternative to (3.5) is

$$Q = a_1(q + yp)^3 + a_5\left(q^2 + \frac{b\lambda - d}{a\mu - c}(qp + pq) + \frac{b\lambda - d}{a\mu - c}\,\frac{b\mu - d}{a\nu - c}p^2\right) + aq + bp, \tag{3.6a}$$

$$P = a_1 x(q + yp)^3 + a_5\left(\lambda q^2 + \mu\frac{b\lambda - d}{a\mu - c}(qp + pq) + \nu\frac{b\lambda - d}{a\mu - c}\,\frac{b\mu - d}{a\nu - c}p^2\right) + cq + dp. \tag{3.6b}$$

4. Fourth-degree canonical transformations

4.1. The product of two second-degree canonical transformations

We saw in § 2 that $T^{(2)}$ transformations are necessarily of the special form (2.1) but the product of two such transformations is *not* of this special form. We can construct at least a certain class of fourth-degree transformations as products of two $T^{(2)}$ transformations. We shall discuss later the interesting question whether this class includes *all* the $T^{(4)}$ transformations. Let us take $T_1^{(2)}$: $(q, p) \to (q', p')$ of the form

$$q' = a_1(q+yp)^2 + aq + bp, \tag{4.1a}$$

$$p' = a_1 x(q+yp)^2 + cq + dp, \tag{4.1b}$$

$$ad - bc = 1, \qquad y = (bx-d)/(ax-c), \tag{4.1c}$$

followed by $T_2^{(2)}$: $(q', p') \to (Q, P)$,

$$Q = a_1'(q'+y'p')^2 + a'q' + b'p', \tag{4.2a}$$

$$P = a_1'x'(q'+y'p')^2 + c'q' + d'p', \tag{4.2b}$$

$$a'd' - b'c' = 1, \qquad y' = (b'x' - d')/(a'x' - c'). \tag{4.2c}$$

The product transformation $T^{(4)} = T_2^{(2)} T_1^{(2)}$: $(q, p) \to (Q, P)$ is

$$\begin{aligned} Q = {} & a_1^2 a_1'(1+xy')^2(q+yp)^4 + 2a_1 a_1'(1+xy')\{(q+yp)^2[(a+y'c)q+(b+y'd)p]\}_{\text{symm}} \\ & + a_1(a'+b'x)(q+yp)^2 + a_1'[(a+y'c)q+(b+y'd)p]^2 \\ & + (aa'+b'c)q + (a'b+b'd)p, \end{aligned} \tag{4.3a}$$

$$\begin{aligned} P = {} & a_1^2 a_1' x'(1+xy')^2(q+yp)^4 + 2a_1 a_1'(1+xy')\{(q+yp)^2[(a+y'c)q+(b+y'd)p]\}_{\text{symm}} \\ & + a_1(c'+d'x)(q+yp)^2 + a_1'x'[(a+y'c)q+(b+y'd)p]^2 \\ & + (ac'+cd')q + (bc'+dd')p, \end{aligned} \tag{4.3b}$$

where the second terms need to be symmetrised.

4.2. The general fourth-degree canonical transformation

We do not know whether every $T^{(4)}$ transformation can be represented by (4.3). Let us follow our procedure for $T^{(2)}$ and $T^{(3)}$ transformations and consider the general $T^{(4)}$ (a_i, b_i) transformation

$$\begin{aligned} Q = {} & a_1 q^4 + 2a_2(q^3p + pq^3) + 3a_3(qpqp + pqpq) + 2a_4(qp^3 + p^3q) + a_5 p^4 + a_6 q^3 \\ & + 3a_7 qpq + 3a_8 pqp + a_9 p^3 + a_{10} q^2 + a_{11}(qp+pq) + a_{12}p^2 + aq + bp, \end{aligned} \tag{4.4a}$$

$$\begin{aligned} P = {} & b_1 q^4 + 2b_2(q^3p + pq^3) + 3b_3(qpqp + pqpq) + 2b_4(qp^3 + p^3q) + b_5 p^4 + b_6 q^3 \\ & + 3b_7 qpq + 3b_8 pqp + b_9 p^3 + b_{10} q^2 + b_{11}(qp+pq) + b_{12}p^2 + cq + dp. \end{aligned} \tag{4.4b}$$

We extend the definitions (3.3) and write

$$\alpha = b_6/a_6, \qquad \beta = b_7/a_7, \qquad \gamma = b_8/a_8, \qquad \delta = b_9/a_9, \tag{4.5a}$$

$$\lambda = b_{10}/a_{10}, \qquad \mu = b_{11}/a_{11}, \qquad \nu = b_{12}/a_{12}; \tag{4.5b}$$

then the condition for canonicity (1.3) requires

$$b_i = a_i x \qquad (i = 1, 2, 3, 4, 5), \tag{4.6a}$$

$$a_2 = a_1 y, \qquad a_3 = a_1 y^2, \qquad a_4 = a_1 y^3, \qquad a_5 = a_1 y^4, \tag{4.6b}$$

$$ad - bc = 1, \tag{4.6c}$$

together with some further equations (Bala 1985) analogous to (3.4*c*, *d*, *e*, *f*) to determine a_6, a_{10}, y, β, γ, δ, μ, ν in terms of a_1, x, α, λ, which may be chosen arbitrarily. In the special case $x = \alpha$ we find that $\beta = \gamma = \delta = x$ and the conditions (4.6) extend (Bala 1985) to give

$$b_j = a_j x \qquad (j = 6, 7, 8, 9), \tag{4.6d}$$

$$a_{11} = a_{10} y (x - \lambda)/(x - \mu), \qquad a_{12} = a_{10} y^2 (x - \lambda)/(x - \nu), \tag{4.6e, f}$$

$$3a_7 y^2 = a_9 + 2y^3 a_6, \qquad 3a_8 y = 2a_9 + y^3 a_6, \tag{4.6g, h}$$

from which it is apparent that the transformation (4.4) reduces to

$$\begin{aligned} Q = {} & a_1 (q + yp)^4 + [(q + yp)^2 (a_6 q + a_9 p/y^2)]_{\text{symm}} \\ & + a_{10} \{q^2 + y[(x - \lambda)/(x - \mu)](qp + pq) + y^2[(x - \lambda)/(x - \nu)]p^2\} + aq + bp, \end{aligned} \tag{4.7a}$$

$$\begin{aligned} P = {} & a_1 x (q + yp)^4 + x[(q + yp)^2 (a_6 q + a_9 p/y^2)]_{\text{symm}} \\ & + a_{10} \{\lambda q^2 + \mu y[(x - \lambda)/(x - \mu)](qp + pq) \\ & + \nu y^2 [(x - \lambda)/(x - \nu)]p^2\} + cq + dp, \end{aligned} \tag{4.7b}$$

where the subscript (symm) denotes that the term should be symmetrised as in (4.3*a*, *b*). On suitably renaming the coefficients the transformation (4.7) may be identified with the product transformation (4.3). Provided $y \neq (bx - d)/(ac - c)$, then

$$x = \alpha (\Leftrightarrow \alpha = \beta = \gamma = \delta) \Leftrightarrow T^{(4)} = T^{(2)} \times T^{(2)}. \tag{4.8}$$

If $y = (bx - d)/(ax - c)$, then $T^{(4)}$ takes the form (2.1).

It should be noted that in addition to the conditions (4.6) already listed, the following must be satisfied (in the case $x = \alpha$):

$$2a_1[y(ax - c) - (bx - d)] + a_{10}(a_9/y^2 - a_6 y)(x - \lambda) = 0, \tag{4.6i}$$

$$y^2 a_6 [2y(ax - c) - 3(bx - d)] + a_9(ax - c) + 4a_{10}^2 y^3 (\mu - \lambda)(x - \lambda)/(x - \mu) = 0, \tag{4.6j}$$

$$\begin{aligned} & y^3 a_6 (bx - d) - a_9[3y(ax - c) - 2(bx - d)] \\ & \quad + 4a_{10}^2 y^4 (\mu - \nu)(x - \lambda)^2/[(x - \mu)(x - \nu)] = 0, \end{aligned} \tag{4.6k}$$

$$y = \frac{b\lambda - d}{a\mu - c} \frac{x - \mu}{x - \lambda} = \frac{b\mu - d}{a\nu - c} \frac{x - \nu}{x - \mu}. \tag{4.6l}$$

These five equations should be solvable to give a_9, a_{10}, y, μ, ν in terms of a_1, a_6, x, λ (or we could fix μ or ν and calculate λ, ν or λ, μ).

Suppose we choose $a_9 = a_6 y^3$; then from (4.6*i*) we see that $a_1 = 0$ provided $y \neq (bx - d)/(ax - c)$. In this case the canonical transformation $T^{(4)}$ degenerates to $T^{(3)}$ and equations (4.6*j*, *k*) become

$$3a_6 (x - \mu)[y(ax - c) - (bx - d)] + 4a_{10}^2 y (\mu - \lambda)(x - \lambda) = 0, \tag{4.9a}$$

$$3a_6 (x - \mu)(x - \nu)[y(ax - c) - (bx - d)] + 4a_{10}^2 y (\nu - \mu)(x - \lambda)^2 = 0. \tag{4.9b}$$

From (4.9a, b) we see that x, λ, μ, ν must be connected by equation (3.4e). Thus on putting $a_9 = a_6 y^3$, equations (4.6d–l) become equations (3.4c–f) with $a_1 \to a_6$, $a_5 \to a_{10}$, $a_6 \to a_{11}$, $a_7 \to a_{12}$. In § 5 we give a numerical example showing how this enables us to construct $T^{(3)}$ transformations with unequal values of x, λ, μ, ν.

5. Third-degree transformations constructed from products of second-degree transformations

We may take the product of any two $T^{(2)}$ transformations to obtain approximate values of the second-degree coefficient ratios λ, μ, ν. These values are systematically recalculated as we shall describe by means of an example.

Let us take

$$T^{(2)}:\ a = 2,\ b = c = d = 1,\ x = 2,\ y = \tfrac{1}{3},\ a_1 = 1, \qquad (5.1a)$$

$$T^{(2)'}:\ a' = b' = c' = 1,\ d' = 2,\ x' = \tfrac{1}{2},\ y = 3,\ a_1' = 3. \qquad (5.1b)$$

Then from (4.3a, b), $T^{(4)} = T^{(2)'} \times T^{(2)}$ has the form

$$Q = 147(q+p/3)^4 + 21[(q+p/3)^2(5q+4p) + (5q+4p)(q+p/3)^2] + 3(5q+4p)^2 + 3(q+p/3)^2 + q + 2p, \qquad (5.2a)$$

$$P = \tfrac{147}{2}(q+p/3)^4 + \tfrac{21}{2}[(q+p/3)^2(5q+4p) + (5q+4p)(q+p/3)^2] + \tfrac{3}{2}(5q+4p)^2 + 5(q+p/3)^2 + 4q + 3p. \qquad (5.2b)$$

Here $\alpha = \beta = \gamma = \delta = x = \frac{1}{2}$ and we read off

$$\lambda = b_{10}/a_{10} = 85/156 = 0.544\,8717, \qquad (5.3a)$$

$$\mu = b_{11}/a_{11} = 95/183 = 0.519\,1256, \qquad (5.3b)$$

$$\nu = b_{12}/a_{12} = 221/435 = 0.508\,0459. \qquad (5.3c)$$

These values of λ, μ, ν satisfy (3.4d) almost exactly; thus with $a = 3$, $b = 2$, $c = 4$, $d = 3$,

$$y = \frac{b\lambda - d}{a\mu - c}\,\frac{x-\mu}{x-\lambda} \Rightarrow y = 0.333\,3325, \qquad (5.4a)$$

$$y = \frac{b\mu - d}{a\nu - c}\,\frac{x-\nu}{x-\mu} \Rightarrow y = 0.333\,3315. \qquad (5.4b)$$

Equation (3.4e) is not satisfied so well, the discrepancy between the two sides of the equation being 0.027%. As discussed in § 3, we may choose x and *one* of the parameters λ, μ, ν arbitrarily, using (3.4d, e) to calculate the others. On taking μ from (5.3b) and eliminating ν we obtain a quadratic equation which yields

$$\lambda = 0.519\,3882 \quad \text{or} \quad 0.518\,9375, \qquad (5.5)$$

from which we calculate the corresponding values of ν and y. Thus we establish the existence of two $T^{(3)}$ transformations with $x = 0.5$, $a = 3$, $b = 2$, $c = 4$, $d = 3$, a_1 arbitrary and

$$\lambda = 0.519\,3882, \quad \mu = 0.519\,1256, \quad \nu = 0.518\,8657, \quad y = 0.7920, \qquad (5.6a)$$

$$\lambda = 0.518\,9375, \quad \mu = 0.519\,1256, \quad \nu = 0.519\,3129, \quad y = 0.8112. \qquad (5.6b)$$

In either case a_5 is determined by equation (3.4f). Although rather close in value, the parameters x, λ, μ, ν are thus seen to be distinct. We note the very appreciable change in the values of y given by (5.4), (5.6).

6. Generating functions

For a canonical transformation $(q, p)\rightarrow(Q, P)$, generating functions $F_1(q, Q)$, $F_2(q, P)$ may be found (Goldstein 1980) such that

$$p=\partial F_1/\partial q, \qquad P=-\partial F_1/\partial Q, \tag{6.1a}$$

$$p=\partial F_2/\partial q, \qquad Q=\partial F_2/\partial P. \tag{6.1b}$$

For the transformation (1.1) we find that

$$F_1=(qQ+Qq-aq^2-dQ^2)/(2b), \tag{6.2a}$$

$$F_2=(qP+Pq+bP^2-cq^2)/(2d). \tag{6.2b}$$

The F_1 generating function has the useful property that for a transformation $T^{(a)}$: $(q, p)\xrightarrow{F_1}(\tilde{q}, \tilde{p})$ with $F_1=F_1^{(a)}(q, \tilde{q})$ followed by $T^{(b)}$: $(\tilde{q}, \tilde{p})\xrightarrow{F_1}(Q, P)$ with $F_1=F_1^{(b)}$ $(\tilde{q}, Q)$, the product transformation $T^{(b)}\times T^{(a)}$: $(q, p)\xrightarrow{F_1}(Q, P)$ has

$$F_1(q, Q)=F_1^{(a)}(q, \tilde{q})+F_1^{(b)}(\tilde{q}, Q). \tag{6.3}$$

In this case when a, b, c, d are functions of the time

$$H(q, p, t)\xrightarrow{F_i(t)}K(Q, P, t)=H(Q, P, t)+\partial F_i/\partial t \qquad (i=1 \text{ or } 2). \tag{6.4}$$

6.1. Second-degree canonical transformations

For simplicity let us consider first the case $x=d/b$, so that $y=0$; then equations (2.7) reduce to

$$Q=a_1q^2+aq+bp, \qquad P=a_1dq^2/b+cq+dp, \tag{6.5a,b}$$

and it is easy to see that

$$F_2(q, P)=(qP+Pq+bP^2-cq^2)/(2d)-\tfrac{1}{3}a_1q^3/b, \tag{6.6}$$

which is a simple extension of (6.2b).

In the case $y\neq0$ we find, as shown in the appendix,

$$F_2(q, P)=\frac{1}{2x}\left(P^2+\frac{(c-ax)d}{a_1xy}P\right)-\frac{1}{2y}\left(q^2+\frac{d}{a_1xy}q\right)$$

$$\pm\frac{c-ax}{12a_1^2y^3}\left(\frac{d^2}{x^2}+\frac{4a_1y^2}{x}P-\frac{4a_1y}{c-ax}q\right)^{3/2}. \tag{6.7}$$

Rather than attempt to use this cumbersome result it is better to take (2.7) as the product of

$$T^{(a)}: \tilde{q}=q+yp, \quad \tilde{p}=p, \tag{6.8a}$$

$$T^{(b)}: Q=a_1\tilde{q}^2+a\tilde{q}+b'\tilde{q}, \quad b'=1/(ax-c),$$

$$P=a_1x\tilde{q}^2+c\tilde{q}+d'\tilde{p}, \quad d'=b'x, \tag{6.8b}$$

for which

$$F_1^{(a)} = (q\tilde{q} + \tilde{q}q - q^2 - \tilde{q}^2)/(2y), \tag{6.9a}$$

$$F_1^{(b)} = (Q\tilde{q} + \tilde{q}Q - a\tilde{q}^2 - d'Q^2)/(2b') - a_1\tilde{q}^3/(3b'). \tag{6.9b}$$

Then the overall generating function is given by (6.3) and, in the case when a, b, c, d or a_1 are functions of the time, the new Hamiltonian is

$$K = H + \partial F_1^{(a)}/\partial t + \partial F_1^{(b)}/\partial t. \tag{6.10}$$

6.2 Third- and fourth-degree canonical transformations

It appears that the only case in which a generating function can be constructed is when $x = \lambda$ (third degree) or $x = \alpha$ (fourth degree). We can split up the transformation in a similar way to (6.8a, b) and use (6.10) to calculate the transformed Hamiltonian.

7. Applications

7.1. Attempt to treat an anharmonic oscillator

In order to obtain the exact solution for an anharmonic oscillator, we need to transform a Hamiltonian of the form

$$H = \sum_{r=3}^{N} \lambda_r q^r + \alpha q^2 + \beta p^2 + \gamma q \qquad (N = 3, 4, 5, \ldots), \tag{7.1}$$

where the coefficients are possibly functions of the time, to a solvable Hamiltonian of the form

$$K = H + \partial F/\partial t = AQ^2 + BP^2 + 2CQP, \tag{7.2}$$

where F is a generating function and A, B, C are suitably chosen constants or functions of the time. Let us suppose that Q, P are related to q, p by a $T^{(3)}$ transformation. The degree of the transformation does not materially affect the argument, but a $T^{(3)}$ gives a more complete picture than a $T^{(2)}$. A general $T^{(3)}$ is difficult to specify explicitly, as we discussed in § 3, and consequently we confine or discussion to the special form (2.1) with $y = 0$:

$$Q = a_1 q^3 + a_2 q^2 + aq + bp + e, \tag{7.3a}$$

$$P = a_1 x q^3 + a_2 x q^2 + cq + dp + f \qquad (x = d/b), \tag{7.3b}$$

with the coefficients possibly time-dependent and connected only by (3.4g). Then from (7.2)

$$H = AQ^2 + BP^2 + 2CQP - \frac{\partial}{\partial t}\left(\frac{1}{d}Pq + \frac{b}{2d}P^2 - \frac{c}{2d}q^2 - \frac{a_1}{4b}q^4 - \frac{a_2}{3b}q^3 + eP - \frac{f}{d}q\right), \tag{7.4}$$

where we have used an obvious extension of (6.6) for the generating function F. Substituting P from (7.3b) into (7.4) after the differentiation, it is easy to see that the coefficients of both q^2p and q^3p vanish if we choose A, B, C, b, d such that

$$2(b^2A + d^2B + 2bdC) - d\dot{b} + b\dot{d} = 0. \tag{7.5}$$

Similarly the coefficient of qp vanishes if

$$2[abA+cdB+(ad+bc)C]+a\dot{d}-c\dot{b}=0 \tag{7.6}$$

and the coefficient of p vanishes if we take $e=f=0$. Furthermore, (7.5) makes the coefficients of q^5 and q^6 vanish (which does not matter) but also makes the coefficient of p^2 vanish. Thus, whether the coefficients are constants or functions of the time, the only Hamiltonian (7.1) that we can transform to (7.2) using (7.3) has the degenerate form

$$H=\lambda q^4+\mu q^3+\alpha q^2+\gamma q+\delta p. \tag{7.7}$$

Flessas (1981a, b) has shown that certain Hamiltonians of the form (7.1) with constant coefficients have exact solutions and it should be possible to apply nonlinear transformations $T^{(n)}$ to discover further exactly solvable systems. With constant coefficients there is no need for a generating function but, as we have seen, the general $T^{(n)}$ transformation ($n\geqslant 3$) is complicated and further work is necessary before we can continue our search for reducible Hamiltonians of the form (7.1).

7.2. *Time-dependent harmonic oscillators*

Constants of the motion for certain time-dependent harmonic oscillators may be discovered by canonically transforming via a suitable $T^{(n)}$ to a new Hamiltonian of a simple form in Q or P. Let us consider as an illustration

$$H=4\,e^{-2t}q^2+(e^{2t}-1)p^2+e^{-t}q, \tag{7.8}$$

which transforms to

$$K=H+\partial F/\partial t=Q, \tag{7.9}$$

where

$$Q=8\,e^{-3t}(q-\tfrac{1}{2}e^{2t}p)^3+4\,e^{-2t}(q-\tfrac{1}{2}e^{2t}p)^2+e^{-t}q. \tag{7.10a}$$

This transformation corresponds to an extension of (2.7) to include third-degree terms. A generating function of the type (6.9) has been used and the coefficients are given by

$$a_1=8\,e^{-3t}, \qquad a_2=4\,e^{-2t}, \qquad a=e^{-t}, \qquad b=c=0, \qquad d=e^{t},$$

$$x=2, \qquad y=-\tfrac{1}{2}e^{2t}. \tag{7.10b}$$

The equations of motion $\dot{Q}=0$, $\dot{P}=-1$ give

$$Q=\text{constant}, \qquad \mathrm{P}=\text{constant}-t \tag{7.11}$$

and provide constants of the motion for (7.8). An unlimited number of examples of this kind may be constructed. Transformations of the type $T^{(4)}=T^{(2)}\times T^{(2)}$ given by (4.3) may be used instead of (2.1). Of course, one has to start with a $T^{(n)}$: $(q,p)\to(Q,P)$ and see what Hamiltonians this will reduce.

8. Conclusion

In § 2 the existence of an important type of nonlinear canonical transformation (2.1) was established. When the function $F(q+yp)$ is taken as a Taylor series, extensions of (2.7) involving higher powers of $(q+yp)$ are obtained. The connections between a, b, c, d and x, y are given by (2.1c, d).

We have considered the general nth-degree transformation $T^n(a, b_i)$ of type (1.2) in the cases $n = 2, 3, 4$ as in (2.4), (3.1), (4.4) respectively. In the case $n = 2$ the situation is very simple: all $T^{(2)}$ transformations are of the type (2.1). For $n = 3$ not all of the transformations are of the type (2.1). The general $T^{(3)}$ has the form (3.5), where we see that the third-degree terms form a perfect cube, as demanded by (2.1), but the second-degree terms no longer necessarily assume the form of a perfect square. The ratios $x = b_1/a_1$ and *one* of $\lambda = b_5/a_5$, $\mu = b_6/a_6$, $\nu = b_7a_7$ can be equal only if *all* of them are equal. Then $y = (bx - d)/(ax - c)$ and we have again the special type (2.1). Our initial attempts to construct numerical examples of solutions of equations (3.4) with unequal x, λ, μ, ν led to complex solutions for x or y. By taking starting values as described in § 5 we were able to find real solutions.

$T^{(4)}$ transformations can be constructed as products of $T^{(2)}$ transformations. Obviously a transformation of the type (2.1),

$$Q = a_1(q + yp)^4 + a_2(q + yp)^3 + a_3(q + yp)^2 + aq + bp, \tag{8.1a}$$

$$P = xQ + (c - ax)(q + yp), \qquad \text{i.e. (2.2}b\text{)} \tag{8.1b}$$

with the constraints

$$ad - bc = 1, \qquad y = (bx - d)/(ax - c), \tag{8.1c}$$

cannot be written in the form (4.3) and does not factorise. We have introduced the parameters α, β, γ, δ and λ, μ, ν given by (4.5a, b) for the terms of third and second degree and we have found that $T^{(4)} = T_2^{(2)} \times T_1^{(2)}$ if, and only if, $x = \alpha = \beta = \gamma = \delta$ and $y \neq (bx - d)/(ax - c)$. In this case the third-degree terms in (4.7a, b) form $(q + yp)^3$ only if we choose $a_q = y^3a_6$. From (4.6i) this is the condition for $a_1 = 0$, i.e. the $T^{(4)}$ degenerates into a $T^{(3)}$. A complete analysis of $T^{(4)}$ transformations leads to conditions to be satisfied (Bala 1985) that are much more complicated than (3.4), as may be seen from (4.6), which constitute a simplified subset. To construct solutions with x, α, β, γ, δ real and unequal is difficult, as it is for $T^{(3)}$ transformations, but we see no reason to suppose that such non-factorisable transformations do not exist. We have established that all $T^{(4)}$ transformations have fourth-degree terms as in (8.1) and similarly for the highest-power terms in other $T^{(n)}$ transformations $n = 5, 6 \ldots$.

$T^{(5)}$ and $T^{(6)}$ transformations are analogous to $T^{(3)}$ and $T^{(4)}$ in that a $T^{(5)}$ is necessarily prime, whereas a $T^{(6)}$ may or may not be factorisable into $T^{(3)} \times T^{(2)}$ (or $T^{(2)} \times T^{(3)}$).

The restricted transformations (2.1) do not enable us to reduce an anharmonic oscillator problem to solvable form. This should be possible in view of the work of Flessas (1981a, b) and Leach (1981). A possible approach would be to reduce H to $K = (xQ - P)^2$, where $(q, p) \to (Q, P)$ is a prime $T^{(n)}$ $(n \geqslant 4)$ with unequal parameters x, α associated with the nth-, $(n-1)$th-degree term.

The set of all transformations $T^{(n)}$ $(n = 2, 3, 4, \ldots)$ obviously forms a group and it is from this point of view that we can expect real progress in a further study of their properties.

Appendix. Generating function for the general second-degree canonical transformation

Differentiating equation (2.2a) and substituting $Q = \partial F_2/\partial P$ according to equation (6.1b), we obtain

$$x\partial F_2/\partial P - P = (ax - c)(q + yp). \tag{A1}$$

Again, eliminating p from equations (2.7a, b), we obtain

$$d\,\partial F_2/\partial P - bP = a_1(d-bx)(q+yp)^2 + q. \tag{A2}$$

Eliminating $q+yp$ from (A1), (A2),

$$a_1(d-bx)(x\partial F_2/\partial P - P)^2 = (ax-c)^2(d\,\partial F_2/\partial P - bP - q). \tag{A3}$$

Let

$$x\,\partial F_2(q,P)/\partial P - P = f(q,P),$$

i.e.

$$\partial F_2/\partial P = (P+f)/x. \tag{A4}$$

Then (A3) gives

$$a_1xyf^2 - d(c-ax)f - (c-ax)[(d-bx)P - xq] = 0. \tag{A5}$$

Solving (A5) and using (A4),

$$\frac{\partial F_2}{\partial P} = \frac{1}{x}P + \frac{c-ax}{2a_1xy}\left[\frac{d}{x} \pm \left(\frac{d^2}{x^2} + \frac{4y}{c-ax}[(d-bx)P - xq]\right)^{1/2}\right]. \tag{A6}$$

Integrating

$$F_2(q,P) = A(q) + \frac{1}{2x}P^2 + \frac{c-ax}{2a_1xy}\left[\frac{d}{x}P \pm \frac{x}{6a_1y^2}\left(\frac{d^2}{x^2} + \frac{4y}{c-ax}[(d-bx)P - xq]\right)^{3/2}\right]. \tag{A7}$$

We start again by eliminating q from (2.7a, b) to obtain

$$p = \partial F_2/\partial q = a_1(c-ax)(q+yp)^2 - c\partial F_2/\partial P + aP. \tag{A8}$$

Eliminating $q+yp$ from (A2) and (A8), we obtain

$$y(d-bx)\,\partial F_2/\partial q = yP - xy\,\partial F_2/\partial P - (d-bx)q. \tag{A9}$$

Now eliminate $\partial F_2/\partial P$ from (A6), (A9) to obtain

$$\frac{\partial F_2}{\partial q} = -\frac{1}{2a_1y^2}\left[\frac{d}{x} \pm \left(\frac{d^2}{x^2} + \frac{4a_1y}{x(c-ax)}[(d-bx)P - qx]\right)^{1/2}\right] - \frac{q}{y}. \tag{A10}$$

Integrating,

$$F_2(q,P) = B(P) - \frac{d}{2a_1xy^2}q - \frac{1}{2y}q^2 \pm \frac{c-ax}{12a_1^2y^3}\left(\frac{d^2}{x^2} + \frac{4a_1y}{x(c-ax)}[(d-bx)P - xq]\right)^{3/2}. \tag{A11}$$

This agrees with (A7) if we take

$$A(q) = -\frac{1}{2y}q^2 - \frac{d}{2a_1xy^2}q, \tag{A12a}$$

$$B(P) = \frac{1}{2x}P^2. \tag{A12b}$$

Thus equation (6.7) has been established.

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